“Trading the volatility skew of the options on the S&P index”

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Contents

1 Introduction and objectives of the work

An option is a financial instrument whose price is derived from another asset, namely the underlying, that has become very popular over the last 30 years. In many situations, both hedgers and speculators find it more attractive to trade a derivative on an asset than to trade the asset itself.
Options prices are generally calculated using the so called Black-Scholes model. Under this model the price of an option depends (among other things) in a estimation of the future standard deviation of the underlying (volatility) . Hence each price has an *implied volatility*. In this document we propose a trading strategy using certain combination of options called vertical spreads. The aim of the strategy is to "monetize" changes in the value of the implied volatility of the options prices. To do so we first studied the *greeks* (see section 2 for an explanation) components of each type of spread, in order to understand the nature of the risks involved in the spreads trading, and specially how changes in implied volatility affect prices (section 3). Before trading any kind of option it is mandatory to analyze the behavior of the volatility estimations, model it, and compare it with the results obtained by other authors, which it is shown in section 4. According to this previous work, in section 5 we propose a trading strategy, which we back tested using historical data. The features of the strategy and the conclusions are discussed in section 6.
In section 2 the concepts related to finance needed to follow the document are briefly described.

2 Basic concepts

2.1 Options general terminology

An option is a contract between two parties: The buyer and the seller. Once the contract is signed the buyer acquires the right (not the obligation) to buy or sell the underlying for a certain *strike price* K to the seller at a certain date in the future namely the *expiry date* or *maturity*.
The buyer is said to be long in the contract, whereas the seller is said to be short in the contract.
There are two basic types of options:

Call option: the holder has the right to buy the underlying asset by a certain date for a certain price.
Put option: the holder has the right to sell the underlying asset by a certain date for a certain price.

There are four different positions when entering an options contract: Long call, short call, long put and short put (see section 2.2).
In this work we study combination of options whose underlying is the S&P 500 mini futures contract. This is a futures contract depending on the S&P 500, which is a capitalization-weighted index of the prices of 500 large-cap common stocks actively traded in the United States. The stocks included in the S&P 500 are those of large publicly held companies that trade on either of the two largest American stock market exchanges: the New York Stock Exchange and the NASDAQ.
Options terminology:

- Spot price S_0, is the current price of the underlying.
- Strike price K, is the price at which currencies are exchanged if the option is exercised.

- Time to expiration T, is the time remaining before the option can be exercised, is expressed in *days until delivery*/360.
- Implied volatility σ, it is an estimation of the standard deviation of the underlying.
- Interest rate r, is the risk-free interest rate.

Let us imaging that today the futures price of the index is 1200 units, and we buy a call option (long call), expiring in 10 days, for a strike price of 1300. If the futures price of the index in ten days is 1350, the buyer would exercise his right to buy the future contract at 1300 units, receiving from the seller (short call) (1350-1300)*N dollars, where N is the nominal value specified in the contract. If at expiration the futures price is below 1300, the buyer does not exercise his right. In this case the sellers earn the price of the option, payed by the buyer when the contract was signed. The position of the option's holder depending on where the spot price is can be:

- **In the money:** strike is more favorable to the holder than the current spot price.
- **At the money:** strike is equal to the current spot price.
- **Out of the money:** strike is less favorable to the holder than the current spot price.

2.2 Payoff

The payoff is the pay that we will receive when the option is exercised, it depends on the position we have on the option and the type of option we have:

- **Call option payoff**: If we have a long position (Figure 1), is the maximum between 0 and the difference between the price of underlying minus the exercise price ($max(0, S - K)$). Otherwise, if we have a short position (Figure 2), is the minimum between 0 and the difference between the exercise price minus the price of underlying ($min(0, K - S)$).

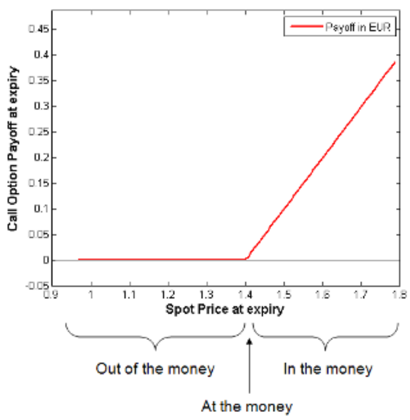

Figure 1: Payoff from long position in a Call option for a strike price $K = 1.4$.

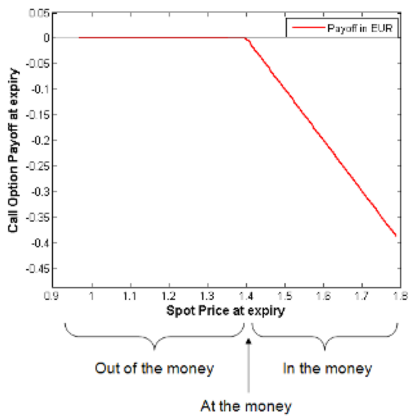

Figure 2: Payoff from short position in a Call option for a strike price $K = 1.4$.

- **Put option payoff**: If we have a long position (Figure 3), is the maximum between 0 and the difference between the exercise price minus the price of underlying ($max(0, K - S)$). Otherwise, if we have a short position (Figure 4), is the minimum between 0 and the difference between the price of underlying minus the exercise price ($min(0, S - K)$).

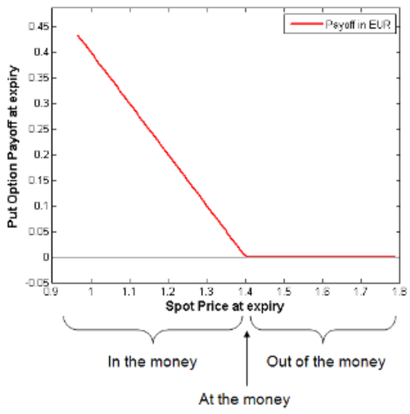

Figure 3: Payoff from long position in a Put option for a strike price $K = 1.4$.

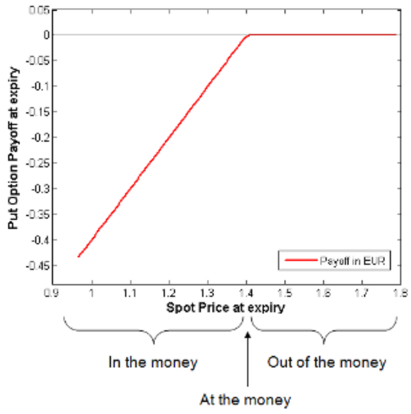

Figure 4: Payoff from short position in a Put option for a strike price $K = 1.4$.

2.3 Option value

The option value is composed by the intrinsic value and the time value. The intrinsic value of an in-the-money option is $S - K$ for calls, and $K - S$ for puts, and the intrinsic value of out of the money options is always 0. The difference between the option value and the intrinsic value is the time value of the option, time value represents the additional value of an option due to the opportunity for the intrinsic value of the option to increase. Figure 5 shows a decomposition of call option value.

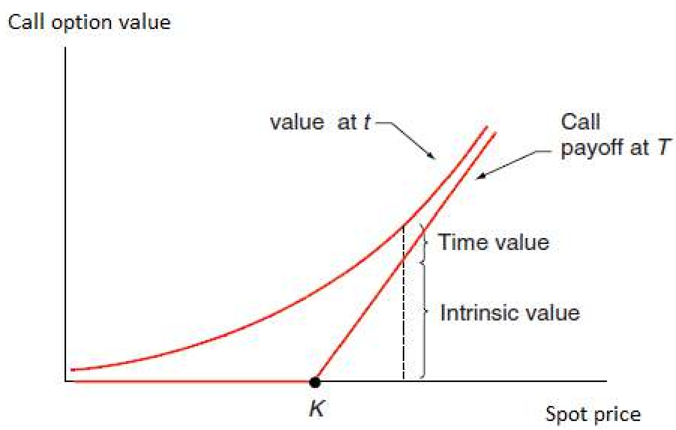

Figure 5: Option value.

2.4 Options pricing

According to the Black-Scholes model, there is a fair value for options, for which no arbitrage conditions through replication can be met. This price can be analytically calculated.
Fisher Black and Myron Scholes received the noble price for developing their scheme. The Black-Scholes model works if the following assumptions are fulfilled:

- The stock price follows a geometric Brownian motion with constant σ and drift.

- There are no restrictions short selling.
- There are no transaction costs.
- All securities are perfectly divisible.
- There are no dividends during the life of the derivative.
- The risk-free rate of interest is constant and the same for all maturities.

Being c a Call options, and p a put option, their prices according to the Black-Scholes model are [3]:

$$c = S_0 e^{-rT}(N(d_1) - KN(d_2)) \tag{1}$$

$$p = Ke^{-rT}(N(-d_2) - S_0 N(-d_1)) \tag{2}$$

where

$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$
$d_2 = d_1 - \sigma\sqrt{T}$

and $N(x)$ is the standard normal cumulative distribution function given by

$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx.$

The price of an option depends on the price of the underlying, the strike price, the risk free rate, and the implied volatility.

2.5 Decomposition of a portfolio profits and losses (P&L) in its greeks components

Suppose that at t_0 we build a portfolio V_0 made of one option, whose price $P(S, t, \sigma, r)$ is obtained using the back scholes model.
When time passes from t_0 to t_1 ($\triangle t = t_1 - t_0$) the portfolio´ s P&L is $\triangle V = V_1 - V_0$, the underlying would have moved $\triangle S = S_1 - S_0$, and the implied volatility $\triangle\sigma = \sigma_1^{imp} - \sigma_0^{imp}$. We can decompose the P&L using a first order Taylor expansion:

$$\triangle P = \delta S + \nu \triangle\sigma^{imp} + \theta\triangle t + \frac{1}{2}\Gamma(\triangle S)^2 \tag{3}$$

Where Delta (δ) is the first derivative of the price with respect to the underlying, Vega (ν) is the first derivative of the price with respect to the implied volatility, Theta (θ) the first derivative of the price with respect to the time to expiration, and Gamma (Γ) the second derivative of the price with respect to the underlying. It can be easily shown that for a geometrical brownian motion $(\triangle S)^2 \approx \triangle t$ [3], therefore all the terms are first order.
δ,ν,Γ and θ are the so called "Greek letters", and they represent the sensitivity of the option price to a single-unit change in the value of either a state variable or a parameter. Such sensitivities can represent the different dimensions to the risk in an option. The greeks of the combination of options are the sum of the individual greeks, and each combination has its own risk profile.
Delta can be interpreted as the probability that an option will end up in the money.

The Theta of an option is defined as the rate of change of the value of the option with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the *time decay* of the option, the change in the price of the option for a 1-day decrease in the time remaining to expiration.
The Vega of an option is defined as the rate of change of the value of the option with respect to the implied volatility of the price.
From equation 1, for a call option, the greeks can be obtained straightforwardly [3].:

$$\begin{aligned} \Delta &= e^{-rT} N(d_1), \\ \Theta &= -\frac{S_0 N'(d_1)}{\sigma\sqrt{T}} - rKe^{-rT} N(d_2) \\ \nu &= S_0 \sqrt{T} N'(d_1) e^{-r_f T} \\ \Gamma &= \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}. \end{aligned}$$

The greeks of a put option can be obtained in the same fashion.
A portfolio λ made of an option minus delta times the underlyng:

$$\lambda = \{O - \delta S\} \tag{4}$$

is said to be delta hedged, since the delta component of the option price is offset by the underlying. This is true as long as the linear approximation implied in equation 3 holds. Usually delta hedged portfolios have to be adjusted one or two time a day, to remain hedged.

3 Vertical spreads

A vertical spread can be defined as the linear combination of long calls and short calls or short puts and long puts, having the long options the same strike price K (long leg) and the short options a different strike price $K\'$ (short leg). The expiration time is the same for all the options that conform the spread. If a vertical spread is conformed by the same number of long and short options, it is called a "pure" vertical spread, otherwise it is a ratio vertical spread. In this document we will study pure vertical spreads, and we will refer to them simply as vertical spreads.
As defined before, a vertical spread has always a long leg and a short leg. At expiration, a vertical spread will have a minimum value of zero, when both options are out of the money (OTM) and a maximum value of the amount between strike prices, when both options are in the money (ITM).According to the strike of each leg, we can classify the vertical spreads into 4 types: bullish call spread, bullish put spread, bearish call spread and bearish put spread [1]. The payoff of every diferent type of spread is obtained by adding the the payoff of its constituent options, and are represented in (Figure6) and (Figure7)

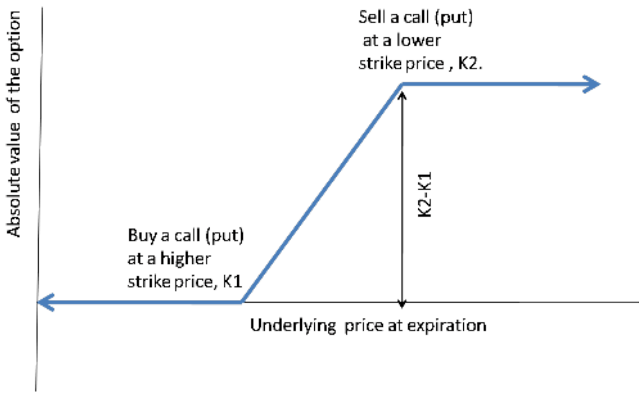

Figure 6: Value of a bullish call or bullish put at expiration

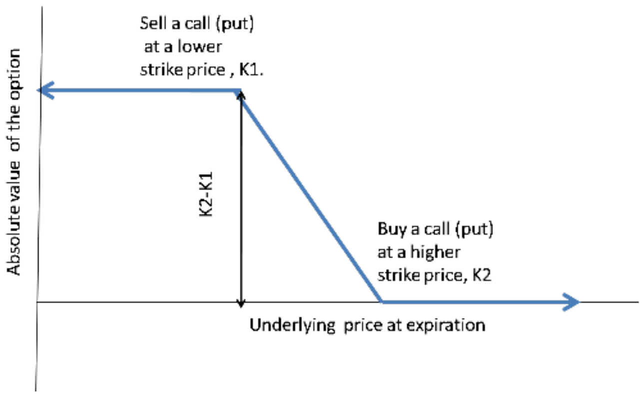

Figure 7: Value of a bearish call or bearish put at expiration

Being $K_{\uparrow}$ and $K_{\downarrow}$ the strikes ($K_{\uparrow} > K_{\downarrow}$), and P_l and P_s the absolute value of the prices of the long and the short leg respectively, the classification can be summarized in Table 1.

Table 1: Different kind of spreads that can be built with the strike prices $K_{\uparrow}$ and $K_{\downarrow}$ being $K_{\uparrow} > K_{\downarrow}$, together with their maximum profits and losses.

	Bullish call spread	Bullish put spread	Bearish call spread	Bearish put spread
Long leg strike	$K_{\downarrow}$	$K_{\downarrow}$	$K_{\uparrow}$	$K_{\uparrow}$
Short leg strike	$K_{\uparrow}$	$K_{\uparrow}$	$K_{\downarrow}$	$K_{\downarrow}$
Maximum prof	$K_{\uparrow} - K_{\downarrow} - (P_l - P_s)$	$P_s - P_l$	$P_s - P_l$	$K_{\uparrow} - K_{\downarrow} - (P_l - P_s)$
Maximum loss	$P_l - P_s$	$K_{\uparrow} - K_{\downarrow} - (P_s - P_l)$	$K_{\uparrow} - K_{\downarrow} - (P_s - P_l)$	$P_l - P_s$

Other authors [2] classify vertical spreads according to wether you get money o give money when you enter in the spread. For example, for a call option, price decreases with strike [3] therefore, entering a bullish call spread costs $P_l - P_s$ whereas entering a bearish call spread gives $P_s - P_l$. In general, regular traders use vertical spreads when they have a directional view on the market and they dont want volatility to be there primary concern [1].

4 Greeks analysis of vertical spreads

In this section, we show the analytical expressions of the greeks for each type of vertical spread, and we analyze their implications. We assume two general strike prices, K1 and K2, where $K2 > K1$, a spot price S, an implied volatility σ, and time to expiration t.

We also show graphically (Figures 8,9,10 and 11) the evolution of the price and the greeks as a function of time and spot price, corresponding to all the different vertical spreads that can be built using the strikes $K1 = 1000$ and $K2 = 1100$. All the calculations were done assuming $\sigma = 30\%$, a risk free interest rate $r = 2.8\%$ and time to expiration 30 days $t = 30/360 = 0.0833$ in agreement with previously recorded market data.

4.1 Bullish call spread.

4.1.1 Delta

Delta, for this type spread, is expressed mathematically as:

$$\delta = e^{-rt}[N(d1) - N(d2)] \tag{5}$$

Where t is time to expiration, $d1 = [ln(S/K1) + \frac{1}{2}\sigma^2 t]/\sigma\sqrt{t}$, $d2 = [ln(S/K2) + \frac{1}{2}\sigma^2 t]/\sigma\sqrt{t}$ and $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. Since $K2 > K1$ then $d1 > d2$ and $N(d1) > N(d2)$, therefore delta is always positive, and the spread always has a bullish perspective.
For $n(d2)/n(d1) = K2/K1$, delta is maximized (this expression can be derived by making the partial derivatives of δ with respect to K1 and K2, and matching both expressions, since at the maximum they are equal to 0).

4.1.2 Gamma

Gamma, for this type of spread, is expressed mathematically as:

$$\Gamma = \frac{1}{S\sigma\sqrt{t}}(n(d1) - n(d2)) \tag{6}$$

Where $n(x)$ is the standardized normal probability density function. Gamma is zero when $n(d1) = n(d2)$. Since the standardized normal density function is symmetrical around 0, the former expression is fulfilled when $d(1) = -d(2)$, which implies that $\sqrt{K1K2} = Se^{\frac{1}{2}\sigma^2 t}$.
When entering the spread, the trader can choose wether to be long or short in gamma adjusting the strike prices of the legs. The highest gamma values are achieved when $n(d1)$ is in its maximum, setting n(d2) as lower as posible. $n(d1)$ is in its maximum when $d1 = 0$ which implies that $K1 = Se^{\frac{1}{2}\sigma^2 t}$. Setting K2 as out of the money as possible minimizes n(d2).Close to the expiring date, gamma "explodes" since t is dividing.

4.1.3 Vega

The analytical expression for vega is

$$\nu = Se^{-rt}\sqrt{t}(n(d1) - n(d2)) \tag{7}$$

Like in the gamma case, vega is zero when $n(d1) = n(d2)$ and $\sqrt{K1K2} = Se^{\frac{1}{2}\sigma^2 t}$. Vega is also maximum for $K1 = Se^{\frac{1}{2}\sigma^2 t}$ and setting $K2$ as higher as possible minimizes n(d2). The further the expiration date, the higher is vega.

4.1.4 "Singular" point

At $\sqrt{K1K2} = Se^{\frac{1}{2}\sigma^2 t}$ gamma, vega and theta are zero. Entering the spread in that position is, for a sufficiently short period of time, equivalent to enter a pure delta position. In the case of $K1 = 1000$, $K2 = 1100$, $r = 2.8\%$, $t = 30/360$ and $\sigma = 30\%$ we obtain $e^{\frac{1}{2}\sigma^2 t} = 1.0003$, therefore $S \approx \sqrt{K1 * K2} = 1048 \approx K1 + \frac{K2-K1}{2}$

4.1.5 Graphs

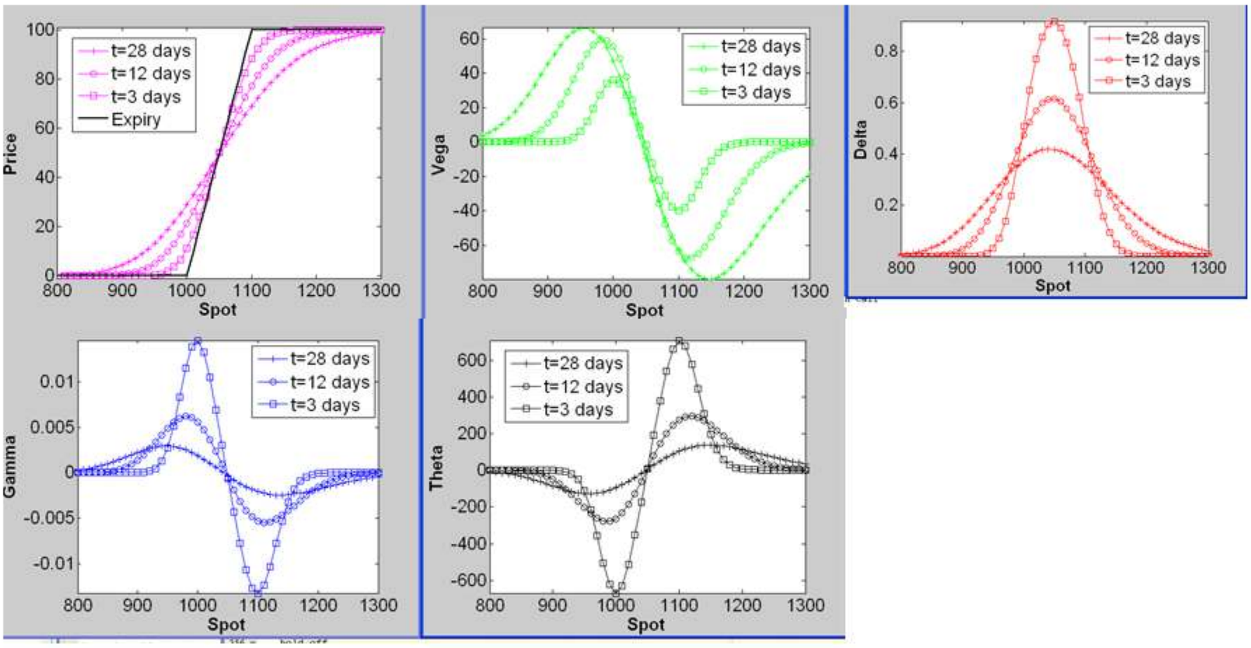

Figure 8: Evolution with time and spot of the price and the greeks of a bullish call spread with short leg strike $K2 = 1100$, long leg strike $K1 = 1000$, $vol = 30\%$ and $r = 2.8$

4.2 Bullish put spread.

The greeks of the bullish put spread are equal to the greeks of the bullish call spread.

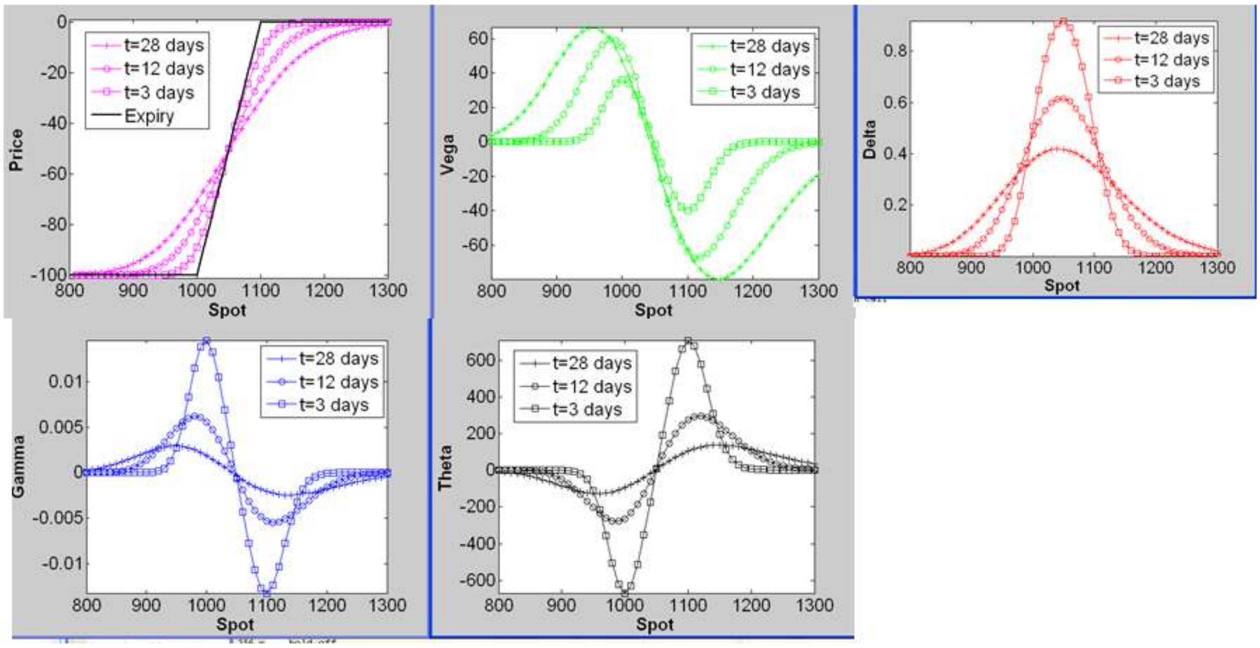

Figure 9: Evolution with time and spot of the price and the greeks of a bullish put spread with short leg strike $K2 = 1100$, long leg strike $K1 = 1000$, $vol = 30\%$ and $r = 2.8$.

4.3 Bearish call spread.

4.3.1 Delta

Delta, for this spread, is expressed mathematically as:

$$\delta = e^{-rt}[N(d2) - N(d1)] \tag{8}$$

Since $K1 < K2$ then $d1 > d2$ and $N(d1) > N(d2)$, therefore delta is always negative, and the spread always has a bearish perspective.
For $n(d2)/n(d1) = K2/K1$, delta is minimized (this expression can be derived by making the partial derivatives of δ with respect to K1 and K2, and matching both expressions, since at the maximum they are equal to 0)).

4.3.2 Gamma

Gamma, for this spread, is expressed mathematically as:

$$\Gamma = \frac{1}{S\sigma\sqrt{t}}(n(d2) - n(d1)) \tag{9}$$

Gamma is zero when $n(d1) = n(d2)$. Since the normal density function is symmetrical around 0, the last expression is fulfilled when $d(1) = -d(2)$ which implies that $\sqrt{K1K2} = Se^{\frac{1}{2}\sigma^2 t}$.
When entering the spread, the trader can choose wether to be long or short in gamma adjusting the strike prices of the legs. The highest gamma values are achieved when $n(d2)$ is in its maximum, setting n(d1) as lower as posible. $n(d2)$ is in its maximum when $d2 = 0$ which implies that $K2 = Se^{\frac{1}{2}\sigma^2 t}$. Setting K1 as much ITM as possible minimizes n(d1). Close to the expirying date, gamma "explodes" since t is dividing.

4.3.3 Vega

The analytical expression for Vega is

$$\nu = Se^{-rt}\sqrt{t}(n(d2) - n(d1)) \tag{10}$$

. Like in the gamma case, Vega is zero when $n(d1) = n(d2)$ and $\sqrt{K1K2} = Se^{\frac{1}{2}\sigma^2 t}$. Vega is also maximum for $K2 = Se^{\frac{1}{2}\sigma^2 t}$ and setting $K1$ as lower as possible minimizes n(d1). Also the further from the exprying date, the higher is gamma.

4.3.4 "Singular" point

At $\sqrt{K1K2} = Se^{\frac{1}{2}\sigma^2 t}$ gamma, vega and theta are zero. Entering the spread in that position is, for a sufficiently short period of time, equivalent to enter a pure delta position.

4.3.5 Graphs

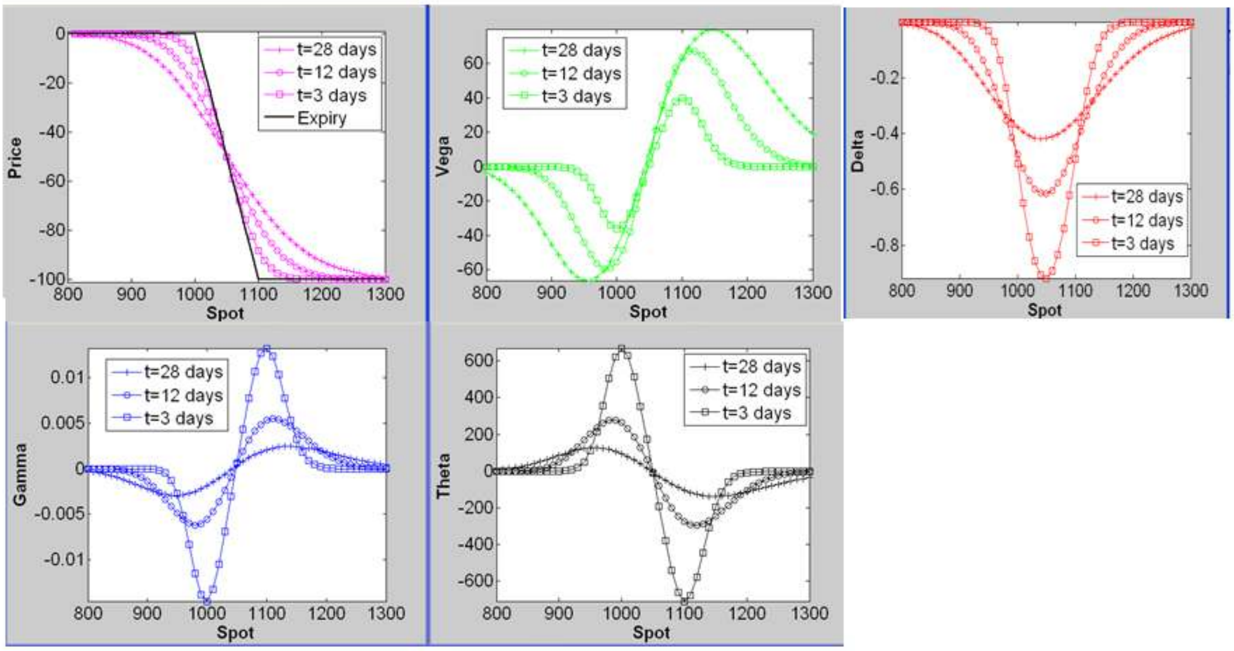

Figure 10: Evolution with time and spot of the price and the greeks of a bearish call spread with short leg strike $K1 = 1000$, long leg strike $K2 = 1100$, $vol = 30\%$ and $r = 2.8$.

4.4 Bearish put spread.

The greeks of a bearish put spread are equivalent to the greeks of bearish call spread.

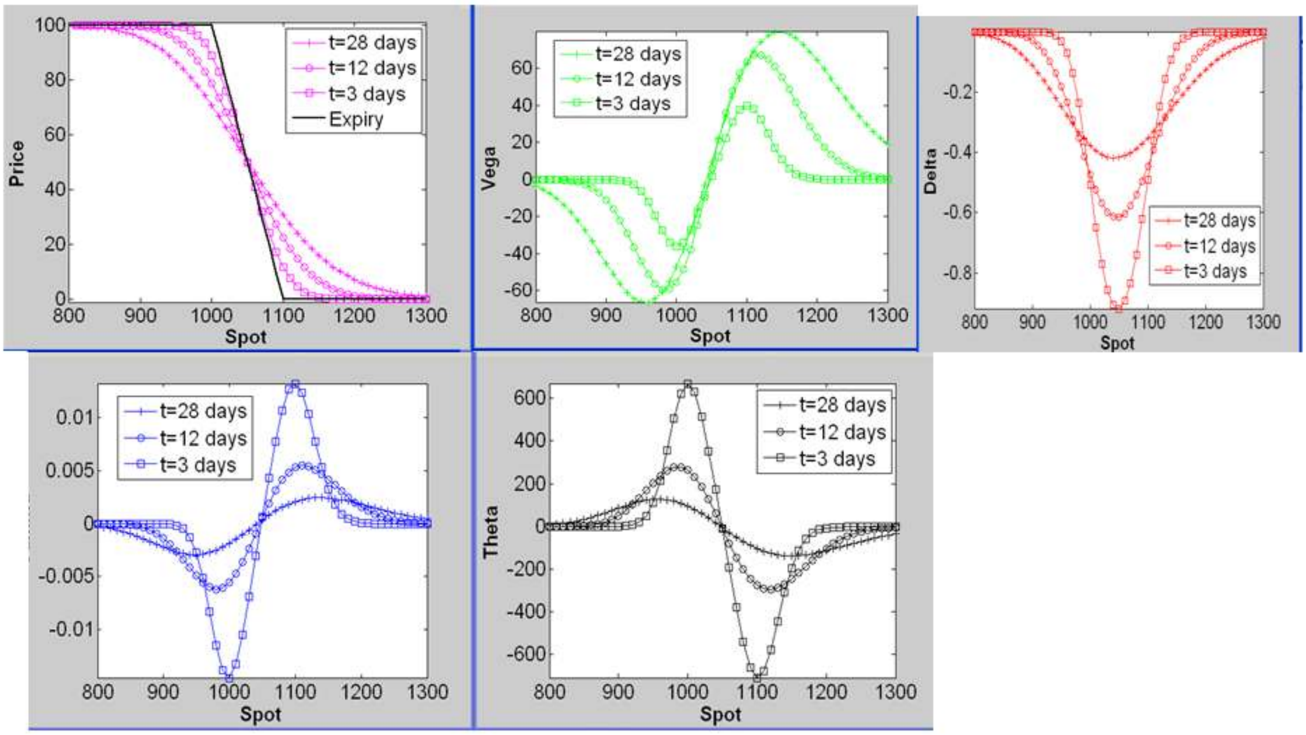

Figure 11: Evolution with time and spot of the price and the greeks of a bearish put spread with short leg strike $K1 = 1000$, long leg strike $K2 = 1100$, $vol = 30\%$ and $r = 2.8$.

4.5 Influence of volatility on the spreads

4.5.1 Low implied volatility scenario

Lets assume that at t_0 we want to enter into a bearish call vertical spread and that the implied volatility of the options is low. We expect that at t_1 the implied volatility suffers an upwards shift of 300 basis points. We have taken the data of two bearish spreads from the data base Bloomberg, and we have calculated how the shift of the implied volatility affect their prices (table 2). Since the aim of the section is to compare the sensibility to changes in implied volatility of 2 similar spreads, we set aside the effects of delta, theta and gamma, by keeping constant t, and S. The days to expiration at t_0 are 18, $t_1 = t_0 + 1$ day, the interest rate is 0.219%, and the spot price is $S = 1187.00$ (S&P 500 mini futures).

Table 2: Evolution of the price of to bearish calls (the price is given in units of the underlying (u), to calculate the price in dollars, the number must be multiplied by the nominal value of the contract), when there is a downwards shift of 3 % of the implied volatility level. For the bearish call 1 the short leg is initially ATM and the long leg is OTM, whereas for the bearish call 2 the short leg is initially ITM and the long leg is ATM

	Short leg strike (K1) and implied vol at t_0	Long leg strike (K2) and implied vol at t_0	Price at t_0	Price at t_1 $\triangle\sigma = 3\%$	P&L
Bearish call 1	K1=ATM $\sigma_i = 29.496\%$	K2=4.21% OTM $\sigma_i = 26.727\%$	-20.75u	-21.31u	-0.56u
Bearish call 2	K1=5%ITM $\sigma_i = 33.565\%$	K2= ATM $\sigma_i = 30.710\%$	-32.83u	-32.33u	0.5u

As we have seen in section 2.3.3, for a bearish call spread, the maximum sensitivity to vega is achieved when the higher strike (long leg) is at $K2 = Se^{\frac{1}{2}\sigma^2 t}$. In this case: $e^{\frac{1}{2}\sigma^2 t} = 1.0002 \approx 1$ hence, the closest is the long leg to the ATM position, the longer in implied volatility is the spread. For these reason, the spread 2 earns money under an hypothetical 3% increment in volatility while the spread 1 does not.

4.5.2 High implied volatility scenario

Using the same parameters as in section 3.6, lets assume that at t_1 the volatility will suffer a downwards shifts of 300 basis points (table 3).

Table 3: Evolution of the price of to bearish calls, when there is a downwards shift of 3 % of the implied volatility level(the price is given in units of the underlying (u), to calculate the price in dollars, the number must be multiplied by the nominal value of the contract). For the bearish call 1 the short leg is initially ATM and the long leg is OTM, whereas for the bearish call 2 the short leg is initially ITM and the long leg is ATM

	Short leg strike (K1) and implied vol at t_0	Long leg strike (K2) and implied vol at t_0	Price at t_0	Price at t_1 $\triangle\sigma = -3\%$	P&L
Bearish call 1	K1=ATM $\sigma_i = 29.496\%$	K2=4.21% OTM $\sigma_i = 26.727\%$	-20.75u	-20.05u	0.7u
Bearish call 2	K1=5%ITM $\sigma_i = 33.565\%$	K2=ATM $\sigma_i = 30.710\%$	-32.83u	-33.10u	-0.27u

As we have seen in section 2.3.3, for a bearish call spread, the maximum sensitivity to vega is achieved when the higher strike (long leg) is at $K2 = Se^{\frac{1}{2}\sigma^2 t}$. In this case: $e^{\frac{1}{2}\sigma^2 t} = 1.0002 \approx 1$, the closest is the long leg to the ATM position, the longer in implied volatility is the spread. For these reason, the spread 1 earns money under an hypothetical 3% decreasing in volatility while the spread 2 does not.

5 The volatility smile

One of the main assumptions of the Black Scholes options pricing model is that the mechanics of the underlying of an option are ruled by a geometrical brownian motion, hence with volatility independent of strike prices, spot prices, or time to expiry (variance goes with Δt in this kind of motion).
The implied volatility in the prices of options prior to October 1987 was in agreement with the former assumption. Until then a plot of the implied vol vs the strike price, or the spot price, showed a straight horizontal line, or a straight horizontal plane in the case of plotting the implied vol vs the strike prices and the time to expiry.
In october 1987 the stock market drop by 20 % or more in two days. This kind of extreme movements conform what is usually called the "heavy tails", and are not taken into account in a geometrical brownian motion model. After this, the markets reacted making more expensive low strike puts than high strike or ATM calls. By doing this, a speculator pays more for a deep OTM strike put, since a extreme favorable movement, with neglige probability in the BSM, is indeed possible. Since then the plot of the implied volatility vs strike prices, resembles a skew or even a smile, depending also on the time to expiration (Figure 12).

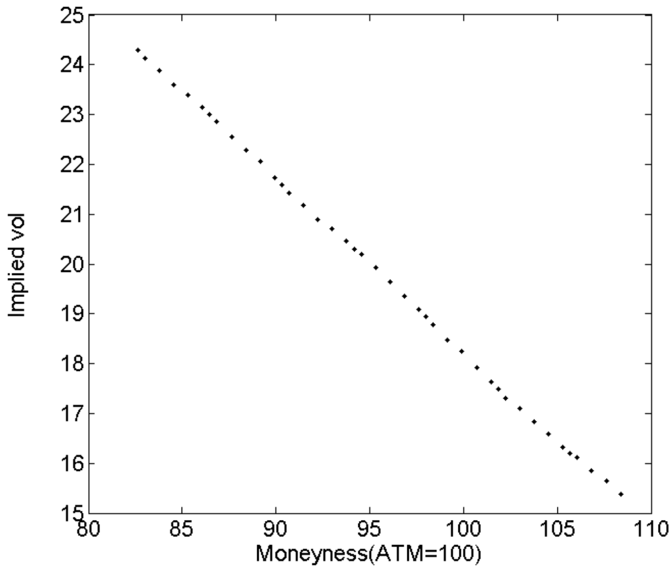

Figure 12: The implied volatility values of the calls on the S&P Index expiring on December 2011 are plot vs the moneynes.We define the moneyness of an option as $m = K/S$ expressed in points per cent, where K is the strike price of the option and S the underlying Spot price. The expiration time is 6 months and two weeks. It is said that the shape is a skew

Each market has developed its own "idiosincratic" volatility smile. Until "the advent" of and alternative model to the BSM, options are priced using BlackScholes, but the implied volatility of the price is adjusted to the volatility smile of the particular option market.
In this section we show the specific properties of the volatility skew associated to the options that we use to make the strategy. To do this we used a set of historical data taken from the database Bloomberg, with the following characteristics: Calls on the S&P 500 mini futures that expire in December 2011. The first sample corresponds to the 1st of june, and the sampling frecuency is 20 min

5.1 Features of the volatility smile of equity index options

In this section we enumerate the general features of the volatility smile according to [7], and we test them using our historical data.

5.1.1 "Volatility are steepest for small expirations as a function of strike, shallower for longer expirations"

This feature can be clearly seen in the historical data.
Let us define the moneyness of an option as $m = K/S$ expressed in points per cent, where K is the strike price of the option and S the underlying Spot price. In figure 13 we show the evolution of the slope (β) of the skew, directly measured from the $m \sim 95\%$ and the $m \sim 90\%$ options using the following expression:

$$\beta(t_i) = \frac{\sigma_{95}(t_i) - \sigma_{90}(t_i)}{(95 - 90)} \tag{11}$$

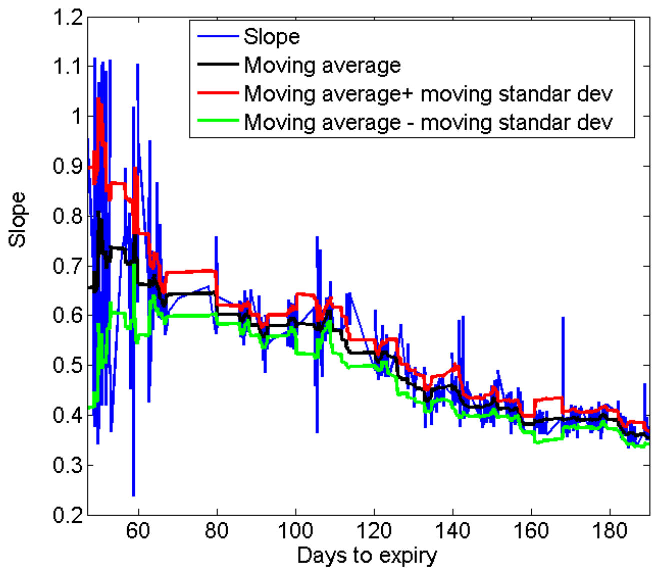

Figure 13: Evolution of the slope (β), calculated according to the equation (7) as function of time to expiry. In the same graph the moving average, the moving average+moving standard deviation and the moving average - moving standard deviation can be found. This statistics were calculated according to equation 12 and equation 13

The moving average was calculated using the following expressions

$$MA(t_i) = \frac{\sum_{n=(i-31)}^{n=i} \beta(t_n)}{30} \tag{12}$$

The standard deviation was calculated using the following expression

$$MS(t_i) = \sqrt{\frac{1}{30}\sum_{n=(i-31)}^{n=i}(\beta(t_n) - MA(t_i))^2} \tag{13}$$

The graph of figure 13 also shows another feature: **The volatility of the volatility is higher the closer is the option to expiration**

5.1.2 "There is a negative correlation between changes in implied ATM volatility and changes in the underlying itself"

In figure 14 we show the "ATM volatility vs time" graph together with the "underlying vs time graph" for our historical data.The negative correlation can be clearly seen.

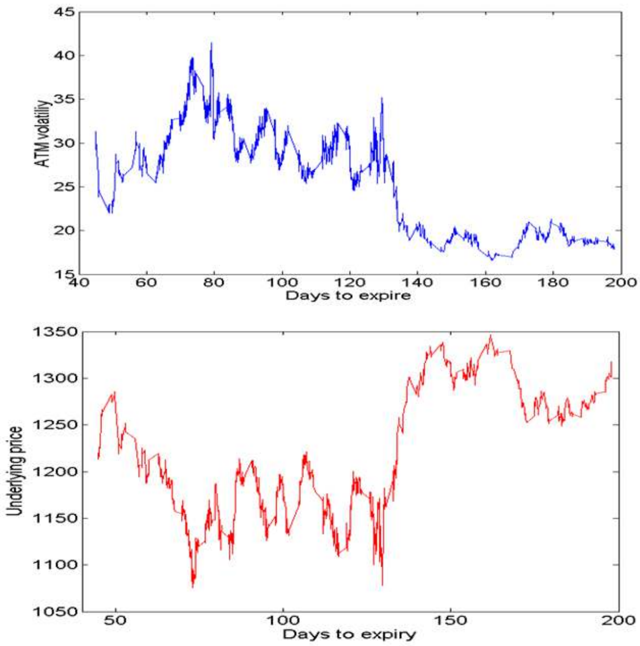

Figure 14: **Up:**Implied volatility of the ATM options, as a function of the time to expiry. **Down:** Spot price of the underlying, as a function of the time to expiry

The value of the Pearson correlation coefficient is

$$\rho = -0.9170 \tag{14}$$

Since index is formed by the sum of n stocks X_i, the variance of the index is given by the expression:

$$var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} var(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} cov(X_i, Xj) \tag{15}$$

When the market drops, the correlation between the stocks increases, increasing also the volatility of the index. For this reason, the implied volatility of the ATM options has a negative correlation with the behavior of the stock.

5.1.3 "Low strike volatilities are usually higher than high-strike volatilities"

In figure 15 we show the implied vols of the Calls expirying on December for different strikes as function of days to expiration. This data is daily data, and was taken from bloomberg. The ITM and OTM labels correspond to Calls, therefore the low strikes correspond to ITM and the high strikes to OTM. For example, if the spot price is 1000, a 5 % ITM strike price would be of 950.

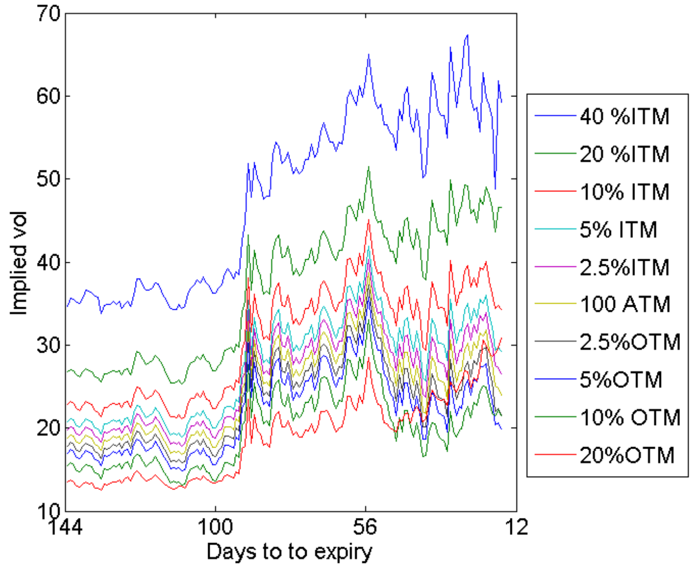

Figure 15: Evolution of the implied volatilities as function of time for different strike prices

The graph clearly shows that the low strikes have higher implied volatilities than the higher strikes.

5.1.4 "After large sudden market declines, the implied volatility of out of the money calls may be greater than for ATM calls, reflecting an expectation that the market will rebound"

We could not see in our high frequency data this behavior. However, the skew showed convexity after large market declines. Although the curvature was not enough for the OTM calls to be higher than the ATM calls, the cause of the curvature is the same: OTM calls get more expensive due to the fact that the market expects a sudden increase in the underlying level (Figure 16).

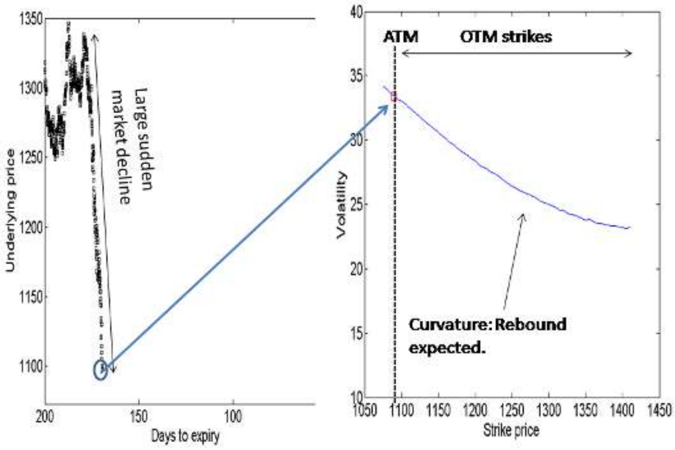

Figure 16: **Left** Plot of the futures price of the S&P 500 mini index, as function of the days to expiry of its associated option. **Right** Plot of the volatility skew of the options at certain time point, that correspond to the last point of the left graph. The point of the skew corresponding to the underlying is highlighted

5.2 Heuristic mathematical modeling of the volatility surface

5.2.1 Sticky delta rule

The sticky delta rule assumes that the implied volatility of an option with a certain time to maturity, depends on the moneyness ($m_t = \frac{K}{S_t}$)of the option. Mathematically this can be expressed as:

$$\sigma_t = f(m_t, T-t) \tag{16}$$

Where T is the expiration date, t the present date and therefore T-t the time to maturity.The following expression is one of the most used models.

$$\sigma(S_t; K, T-t) - \sigma(S_t, S_t, T-t) = \alpha_0 + \alpha_1 ln(m_t) + \alpha_2 [ln(m_t)]^2 + \alpha_3 (T-t) + \alpha_4 (T-t)^2 + \alpha_5 ln(m_t)(T-t) + \epsilon_t(K, T-t) \tag{17}$$

Where $\sigma(S_t, S_t, T-t)$ is the ATM volatility and $\sigma(S_t; K, T-t)$ the volatility for the option with strike price K.

5.2.2 Applying the model for our historical data

After applying the model to our historical using linear regression we get the following statistics:

$$msf = 0.2618 \tag{18}$$

$$R^2 = 0.9755 \tag{19}$$

In figure 17 we show the surface formed by all the skews along the time given by the model and given by the real data.

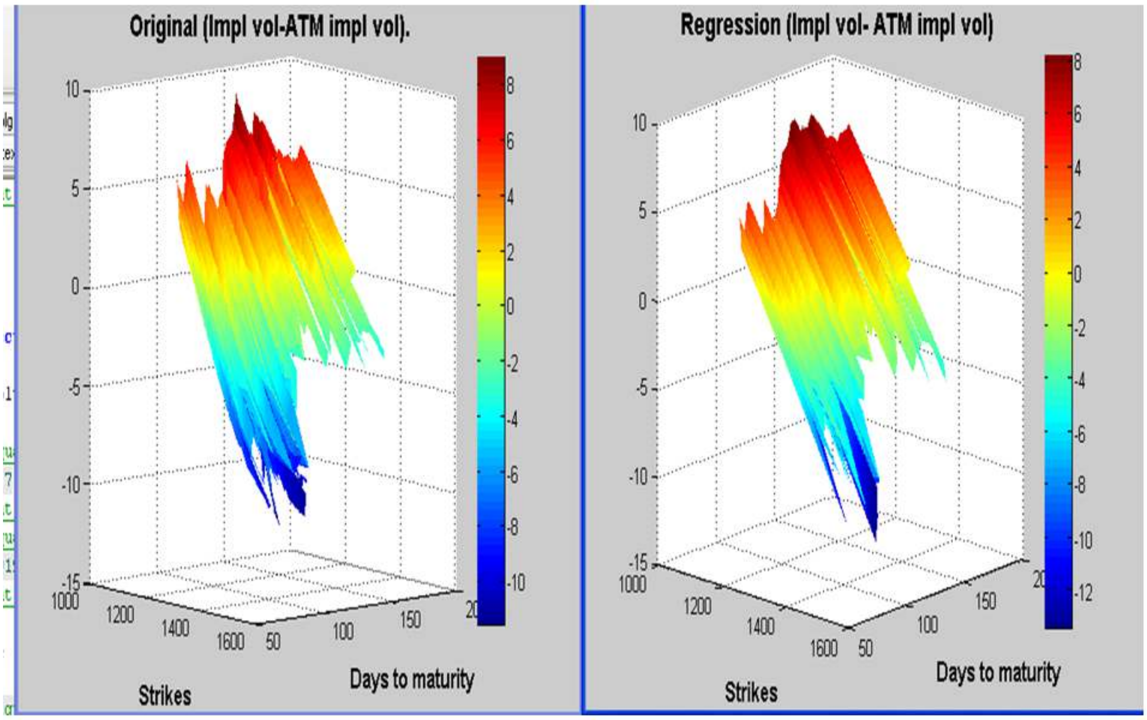

Figure 17: **Left** Actual implied volatility- ATM implied volatility as function of strike prices and days to maturity.**Right**implied volatility- ATM implied volatility as function of strike prices and days to maturity, given by the sticky delta rule

Table 4: Coefficients given by the linear regression corresponding to equation 17,together with their standard error

Variables	Coefficient	Standard error
constant	0.001692	0.00093
$ln(m)$	-0.795201	0.0041
$ln(m)^2$	0.036082	0.0091
$(T-t)$	-0.002	0.000253
$(T-t)^2$	0.0001	0.000016
$(T-t)*ln(m)$	0.0395	0.000517

5.2.3 Sticky strike rule

The sticky strike rule assumes that the implied volatility of an option with a certain strike remains constant when the asset price moves. Mathematically this can be expressed as:

$$\sigma_t = f(K, T - t) \tag{20}$$

One of the most used models follows this expression:

$$\sigma(K, T-t) = \alpha_0 + \alpha_1 ln(K) + \alpha_2 [ln(K)]^2 + \alpha_3 (T-t) + \alpha_4 (T-t)^2 + \alpha_5 ln(K)(T-t) + \epsilon_t(K, T-t) \tag{21}$$

5.2.4 Applying the model for our historical data

After applying the model to our historical data, using a linear regression we get the following statistics:

$$msf = 1.7337 \tag{22}$$

$$R^2 = 0.9195 \tag{23}$$

In figure 18 we show the surface formed by all the skews along the time given by the model and given by the real data.

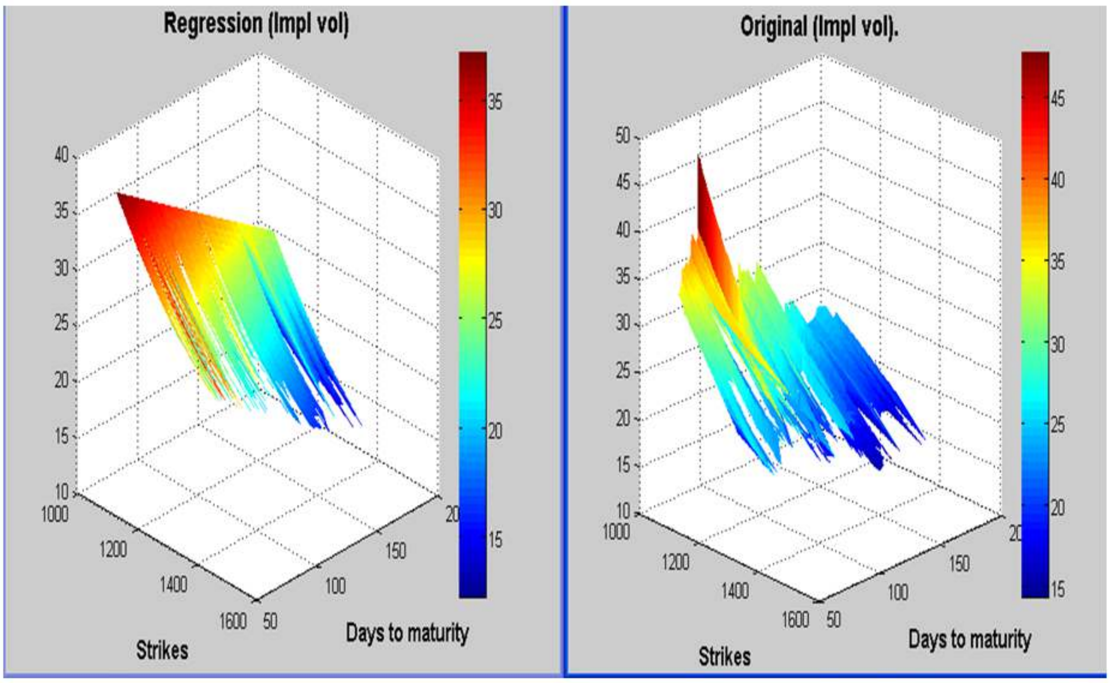

Figure 18: **Left** Actual volatility surface **Right** Volatility surface given by the sticky strike rule

Table 5: Coefficients given by the linear regression corresponding to equation 17,together with their standard error

Variables	Coefficient	Standard error
constant	13.6087	1.6561
$ln(K)$	-2.9178	0.4674
$ln(K)^2$	0.1498	0.0330
$(T-t)$	-0.3038	0.0084
$(T-t)^2$	0.0012	0.00004
$(T-t)*ln(K)$	0.0379	0.0012

5.3 Comparison with other authors results

5.3.1 Sticky delta rule

In the following table we show the results found in [5] and [6] together with ours.

author	Variables	Coefficient	Standar error
Romo	constant	0.0051	0.0004
Daglish		0.0058	0.0004
Aguirre		0.001692	0.00093
Romo	$ln(m)$	-0.2429	0.0025
Daglish		-0.2884075	0.0025
Aguirre		-0.795201	0.0041
Romo	$ln(m)^2$	0.1006	0.0069
Daglish		0.0000000	0.000000
Aguirre		0.036082	0.0091
Romo	$(T-t)$	-0.0045	0.0004
Daglish		-0.007574	0.0004
Aguirre		-0.002	0.000253
Romo	$(T-t)^2$	0.0008	0.0001
Daglish		0.0015705	0.0001
Aguirre		0.0001	0.000016
Romo	$(T-t)ln(m)$	0.0338	0.0008
Daglish		0.0414902	0.0008
Aguirre		0.0395	0.000517

Clearly our results resemble the ones presented in other studies. Obviously, the underlyng and the options are the same in the 3 studies. The difference is in the samples. The sample period of the study presented in [5] is February 2004 to October 2007, and the number of observations: 1575). The sample period presented in [6]is from June 1998 to April 2002, and the total number of observations is 1974. In our case the sample period is from June 2011 to November 2011, and the total number of observations is 100.
In [5] we found, $R^2 = 0.9769$, whereas in [6]is $R^2 = 0.9493$, being $R^2 = 0.9901$ our result.

5.3.2 Sticky strike rule

author	Variables	Coefficient	Standar error
Romo Aguirre	constant	11.9878 13.6087	0.7683 1.6561
Romo Aguirre	$ln(K)$	-2.4985 -2.9178	0.1650 0.4674
Romo Aguirre	$ln(K)^2$	0.1317 0.1498	0.0089 0.0330
Romo Aguirre	$(T-t)$	-0.0590 -0.3038	0.0168 0.0084
Romo Aguirre	$(T-t)^2$	-0.0011 0.0012	0.0005 0.00004
Romo Aguirre	$(T-t)ln(K)$	0.0078 0.0379	0.0005 0.0012

The result found in [6] cannot be compared with the ones of [5] and ours, since his model does not use logarithm. The sticky strike model in this case is :

$$\sigma(K, T-t) = \alpha_0 + \alpha_1 K + \alpha_2 [K]^2 + \alpha_3 (T-t) + \alpha_4 (T-t)^2 + \epsilon_t (K, T-t) \tag{24}$$

Variables	Coefficient	Standar error
constant	0.4438616	0.0238337
$ln(m)$	-0.0001944	0.000036
$ln(m)^2$	0.0000000	0.000000
$(T-t)$	-0.0262681	0.0033577
$(T-t)^2$	-0.0006589	0.0003454
$(T-t)ln(m)$	0.0000290	0.0000000022

$R^2 = 0.9493$

6 Trading the slope

In figure 19 we show the volatility skew of the calls on the S&P 500 mini Futures at a certain time point. The graph sketches the implied volatility vs moneyness. (We define the moneyness of an option as $m = K/S$ expressed in points per cent, where K is the strike price of the option and S the underlying Spot price).

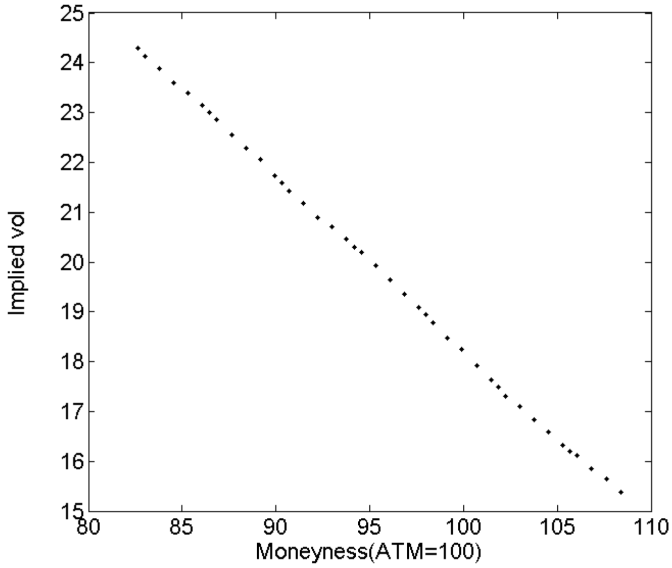

Figure 19: Volatility skew for the calls expiring on December 2011. The expiration time is 6 months and two weeks. The strike price closest to the at the money value is 1300, being 1301.50 the underlying price.

For measuring the slope β_t "directly" we use the following formula:

$$\beta(t_i) = \frac{\sigma_{m_1}(t_i) - \sigma_{m_2}(t_i)}{m_1 - m_2} \tag{25}$$

In figure 13 we showed the evolution of β calculated from the intraday historical data together witgh the moving average and the moving standar deviation
The change in the slope during one day can be expressed as:

$$\triangle\beta(t_i) = \beta(t_i + 1day) - \beta(t_i) \tag{26}$$

In figure 20 we can see that the evolution of $\triangle\beta(t_i)$, using the the data presented in figure 13, is mean reverting, therefore, we could build a trading strategy based on trading changes in the slope of the skew.

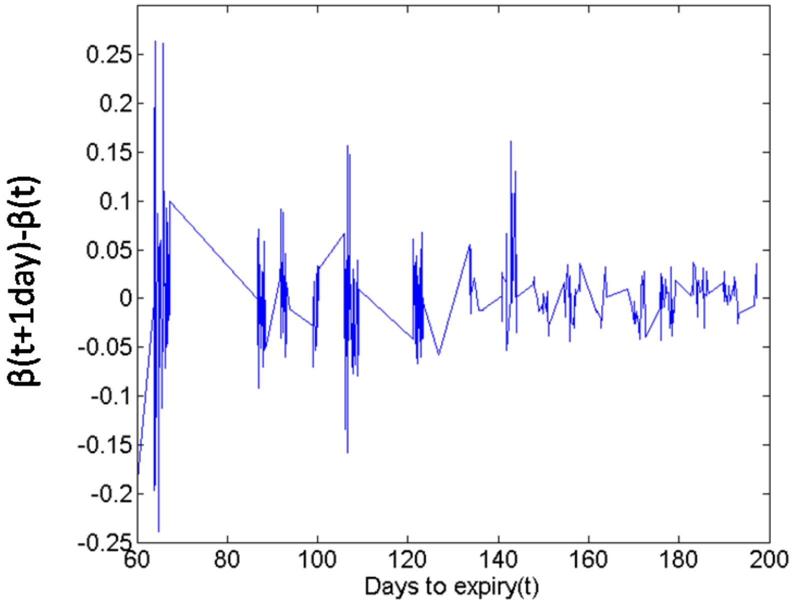

Figure 20: Plot of the one day difference in slope (calculated using equation 1), as a function of the days to expiry

6.1 How can we monetize changes in the slope of the skew?

Let us focus in two points of the volatility skew, with implied volatility σ_1 and σ_2, corresponding to the strike prices K1 and K2 at t_0, and t_1. Let us assume that the moneyness of the options remains constant. Let us assume also that we will deal always with ITM calls. This is done for two reasons: The ITM calls are actively traded and the ITM part of the skew is the part that best fulfills the heuristic models.
At t_0, the slope is: $\beta(t_0) = \frac{\sigma_1(t_0)-\sigma_2(t_0)}{m_1-m_2}$. Lets make the naive assumption that since $\triangle\beta(t)$ is mean reverting, we can find a time point t_1 where the slope raises ($\beta(t_1) - \beta(t_0) > 0$).If this happens, the following inequality must be fulfilled:

$$\sigma_1(t_0) - \sigma_2(t_0) < \sigma_1(t_1) - \sigma_2(t_1) \tag{27}$$

In other words, the distance between the two implied volatility values should increase. In figure 21 we show this graphically (we assume that there is not a big change in the level of the skew)

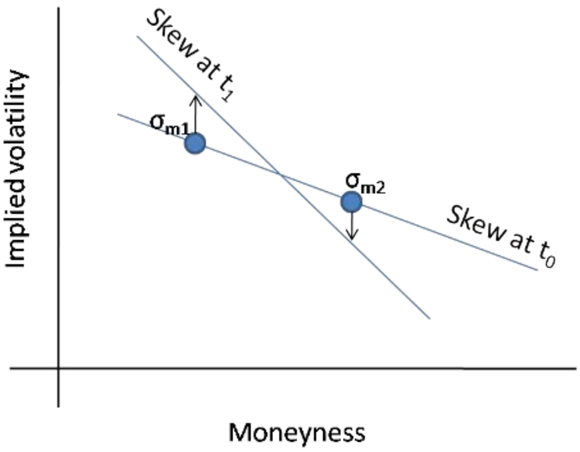

Figure 21: Representation of a change in slope between t_0 and t_1. In this case there is not a big change in the level of the skew

In other to monetize this "movement", we can build a portfolio made of a long call with strike price K1, and short call with strike price K2, being $K2 > K1$. The long call is long in implied volatility whereas the short call is short in implied volatility, monetizing the slope movement. This portfolio is a bullish call. If we expect the slope of the skew to decrease, the monetization would come from building a portfolio consisting in a short call at K1, and a long call at K2, ending up with a bullish call. This positions have the usual risks related to options:

- Delta risk: The portfolio has an overall exposure to delta that has to be taken into account. Delta hedging the portfolio can overcame this risk. The deeper ITM are both options the lowest delta exposure (see section 4).
- Gamma risk: The portfolio has an overall exposure to gamma that has to be taken into account.Provided that the options are ITM calls, gamma is positive for the bearish call, not representing a risk. For the bullish call, gamma is negative, however, choosing the options sufficiently far away from expiration, and holding the positon for a short period of time can overcome this risk (see section 4)
- Theta risk: The portfolio has an overall exposure to theta that has to be taken into account. The behavior of theta is antisimetrical to the behaviour of gamma. Choosing the options sufficiently far away from expiration (see section 4), can overcome this risk.
- Vega risk: Big changes in the level of the skew can lead to a negative P&L of the portfolio, even though the expected change in slope happens (see figure 22). This is due to the fact that the option closer to the ATM point has a bigger vega sensitivity. This risk could be hedged building a ratio vertical spread, instead of a pure vertical spread. The proportion of each leg has to be set to make the vega of the hole portfolio zero. By doing this, changes in the distance between the implied volatility of each strike (i.e changes in slope), are still monetize, and huge level variation does not a affect the price.

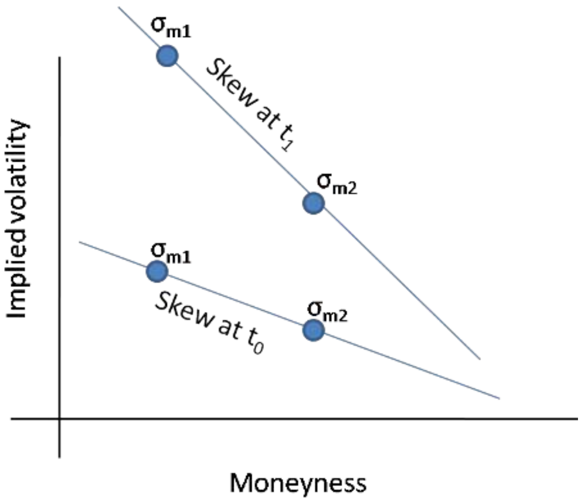

Figure 22: Representation of a change in slope between t^0 and t^1. In this case there is a big change in the level of the skew. A portfolio long in σ_0 and short in $\sigma 1$ would monetize the slope movement. However, if the vega of the option corresponding to σ_1 is higher than the vega of he option corresponding to $\sigma 0$, the whole portfolio may loose money

6.2 An example of a trade

In this subsection we show an example of the trade that will be repeated while doing the strategy, taken from real market data. In this case we monetize a decreasing of the slope. In figure 23 we show the volatility skew at t_0 and at t_1 .

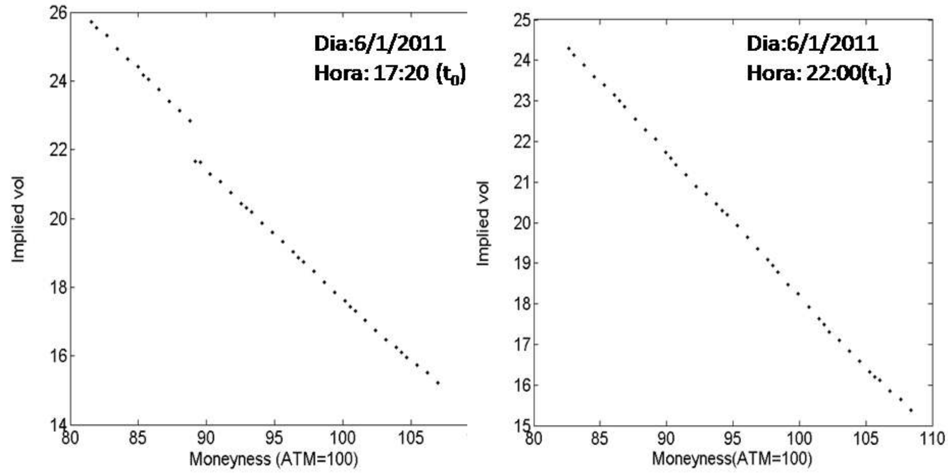

Figure 23: Volatility skew at two different times, corresponding to the calls expiring on December 2011.

At t_0 we can see that near to m=90 there is a jump in the skew, which implies the the slope is "locally high". After time passes, the price of the option is corrected and so the slope.

6.2.1 The trade

The bearish call described in Table 6 can monetize the eventual move of the skew. (All the prices and P&L are given in units of the underlying. To obtain its value in dollar, it must be multiplied by the nominal value of the contract)

Table 6: Legs of the spread used to monetize he change in implied volatility at t_0

Days to expiry		Short Leg	Long leg	Spread
199	Implied vol	23.34%	20.77%	
	Spot	1298.75 u	1298.75 u	
	Strike	1160	1210	
	Price	172.5 u	130.25 u	-42.25 u
	Delta	-0.77	0.71	-0.065
	Gamma	-0.0013	0.0017	0.00037
	Vega	-291.39	331.85	40.45
	Theta	63.70	-73.64	-9.93656

We build a delta hedged portfolio

$$Portfolio_{t0} = bearishcall_{t0} + 0.0651 * underlying_{t0} = -42.25 + 84.5486 = 42.2986\$. \qquad (28)$$

The spread price at t_1 is shown in table 7.

Table 7: Legs of the spread used to monetize he change in implied volatility at t_0

Days to expiry		Short Leg	Long leg	Spread
199	Implied vol	21.74%	20.35%	
	Spot	1301.50 u	1301.50 u	
	Strike	1160	1210	
	Price	162.75 u	124.25 u	-38.5 u

the portfolio is now composed of:

$$Portfolio_{t1} = bearishcall_{t1} + 0.0651 * underlying_{t1} = -38.5 + 84.7276 = 46.22765u \qquad (29)$$

If we undo the position at t_1, we get the following P&L.

$$P\&L = 3.9291u \qquad (30)$$

The 160 basis points decreasing of the implied volatility of the short leg, generates enough earnings to overcome the losses due to the 42 basis points decreasing of the implied volatility of the long leg. 118 volatility basis points have been converted in nearly 4 units.

6.2.2 Greeks decomposition of the P&L

Table 8: Greeks decompostions of P&L of the portfolio used to monetize he change in implied volatility between t_0 and t_1

	Short Leg	Long leg	Spread
Vega P&L	4.67 u	-1.11 u	3.55 u
Gamma P&L	-0.0055 u	0.0071 u	0.0015 u
Theta P&L	0.18 u	-0.20 u	-0.028 u
Greeks total P&L (delta hedged) 3.53 u			
Actual total P&L (delta hedged) 3.9291 u			

Due to the long distance to expiry (half year), the vega contribution is much higher than the gamma or the theta contributions.
Also, the underlying barely moves $\triangle S = 2.75$ therefore the contribution of gamma is nearly zero. Table 9 shows the greeks P&L decomposition if the trade would have been done, non delta hedge

Table 9: Greeks decompostions of P&L of the portfolio used to monetize he change in implied volatility between t_0 and t_1

	Short Leg	Long leg	Spread
Delta P&L	-2.22 u	2.028 u	-0.187 u
Vega P&L	4.67 u	-1.11 u	3.55 u
Gamma P&L	-0.0055 u	0.0071 u	0.0015 u
Theta P&L	0.18 u	-0.20 u	-0.028 u
Greeks total P&L (non delta hedged) 3.34 u			
Actual total P&L (non delta hedged) 3.75 u			

6.3 Systematization of the trading strategy

Let us systematize the trading strategy, trying to monetize increments in the slope using similar trades as the one presented in section 6.2.
We calculate the slope applying equation 25 on the historical data for three cases:

- Strategy A $m_1 \sim 90$ and $m_2 \sim 95$
- Strategy B $m_1 \sim 85$ and $m_2 \sim 90$
- Strategy C $m_1 \sim 80$ and $m_2 \sim 85$

6.3.1 Entry rule

For every case we entry a trade at every time point i where the following inequality is fulfilled:

$$\beta(t_i) < MA(t_i) - MS(t_i). \tag{31}$$

Therefore we expect the slope to raise. To monetize the eventual move we build the following portfolio:

$$P = \{C_{k1} - C_{k2}\} \tag{32}$$

were $k_1 = m_1 * S$ and $k_2 = m_2 * S$, being S the underlying. We also perform the strategy delta hedging the portfolio. In this case, the position is

$$P = \{C_{k_1} - C_{k_2} + (|\delta_{k2}| - |\delta_{k1}|) * S\} \tag{33}$$

6.3.2 Exit rule

We exit a trade as soon as we reach a time point j that satisfies:

$$\beta(t_j) \geq MA(t_j). \tag{34}$$

The positions remains open while $t_j - t_i < 1$day. If equation 34 is not fulfilled after one day, we exit the trade assuming the eventual losses.

6.4 Results

6.4.1 Strategy A, non delta hedged

In figure 24 we show the P&L of each of the built portfolios, together with the greeks approximation .

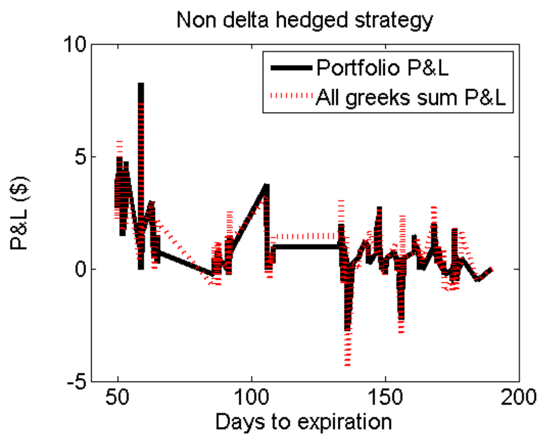

Figure 24: Total P&L, together with the summed greeks components, for every traded day. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

The total number of trades performed were 165, from which 124 were winners.

In figure 25 we show the accrued P&L.

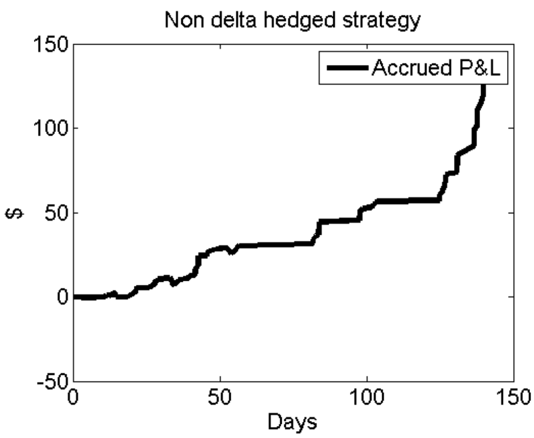

Figure 25: The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract. The x axis corresponds to every day where a position is opened and closed

For each trade, we have calculated the contribution of each greek. This is shown in figure 26.

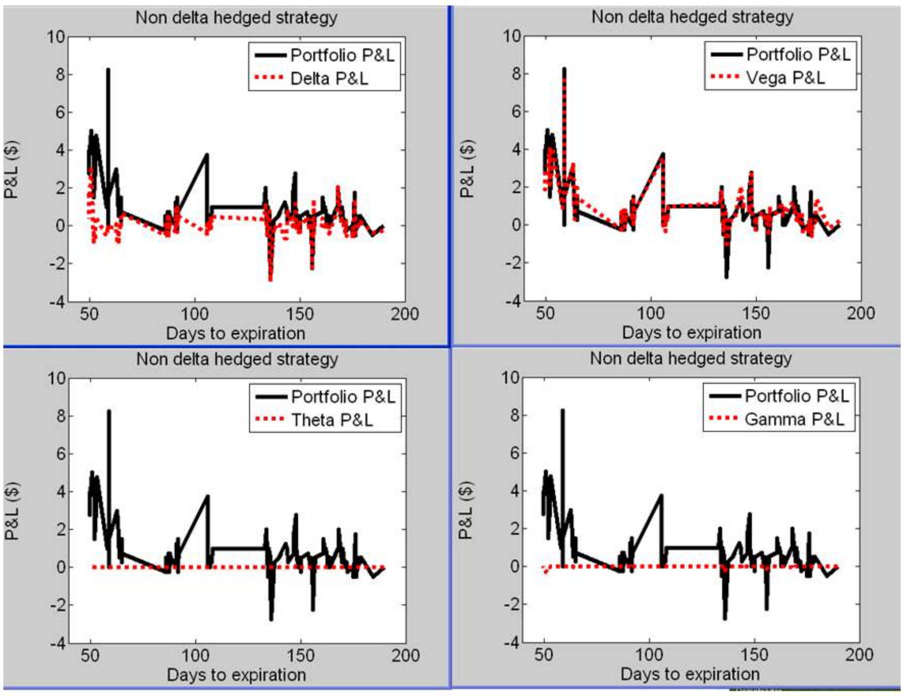

Figure 26: Contribution of each greek to the P&L of the portfolio, together with the total P&L. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

6.4.2 Strategy A, delta hedged

In figure 27 we show the P&L of each of the built portfolios, together with the greeks approximation .

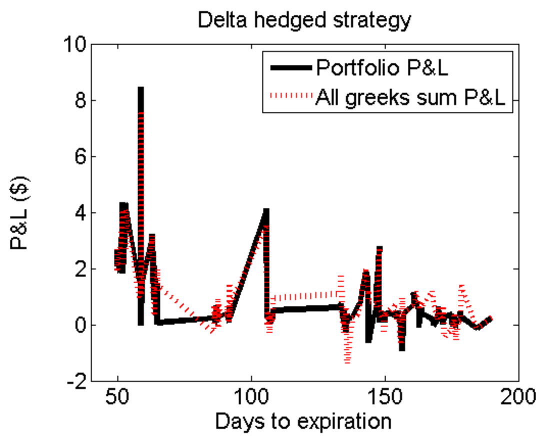

Figure 27: Total P&L, together with the summed greeks components, for every traded day. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

The total number of trades performed were 165, from which 153 were winners.

In figure 28 we show the accrued P&L.

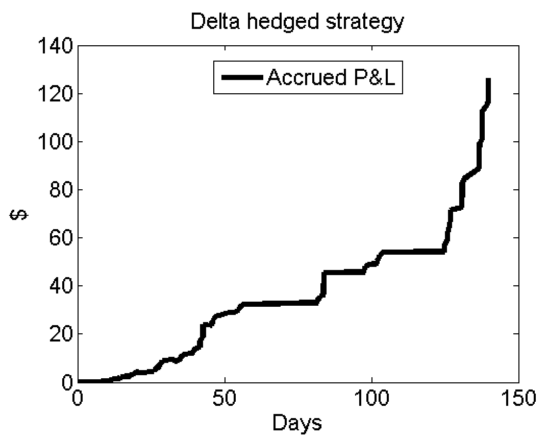

Figure 28: The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract. The x axis corresponds to every day where a position is opened and closed

For each trade, we have calculated the contribution of each greek. This is shown in figure 29.

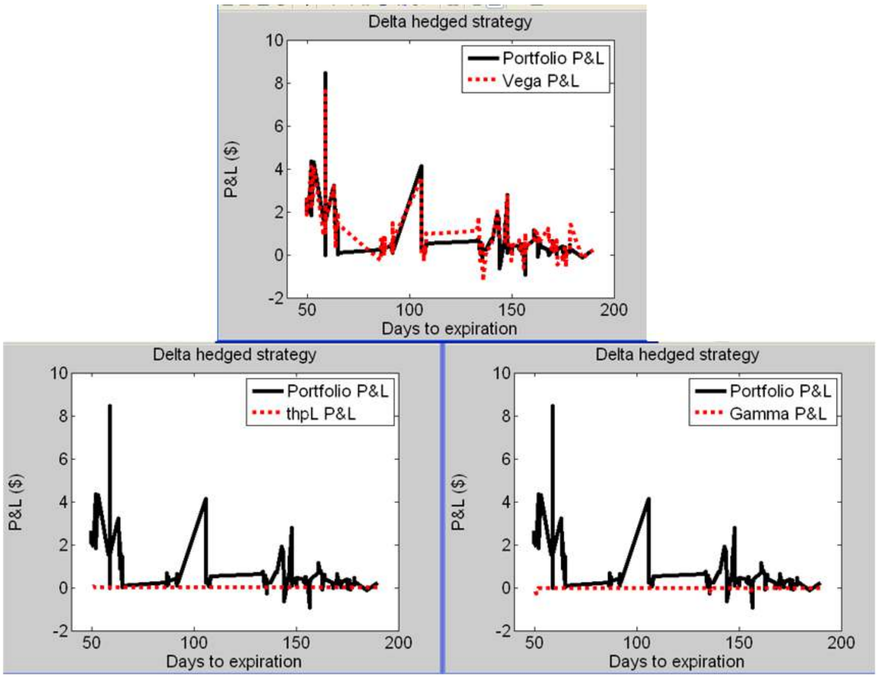

Figure 29: Contribution of each greek to the P&L of the portfolio, together with the total P&L. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

6.4.3 Strategy B, not delta hedged

In figure 30 we show the P&L of each of the built portfolios, together with the greeks approximation .

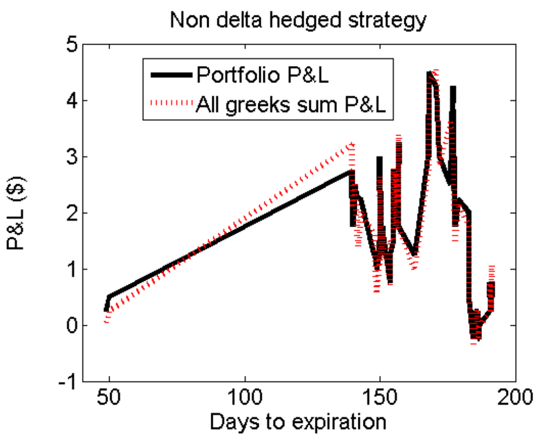

Figure 30: Total P&L, together with the summed greeks components, for every traded day. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

The total number of trades performed were 57 , from which 53 were winners.

In figure 31 we show the accrued P&L.

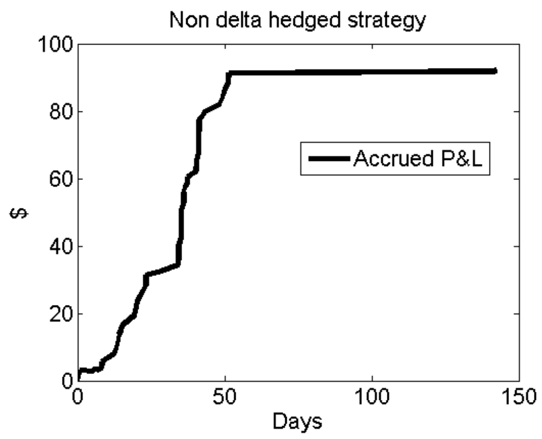

Figure 31: The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract. The x axis corresponds to every day where a position is opened and closed

For each trade, we have calculated the contribution of each greek. This is shown in figure 32.

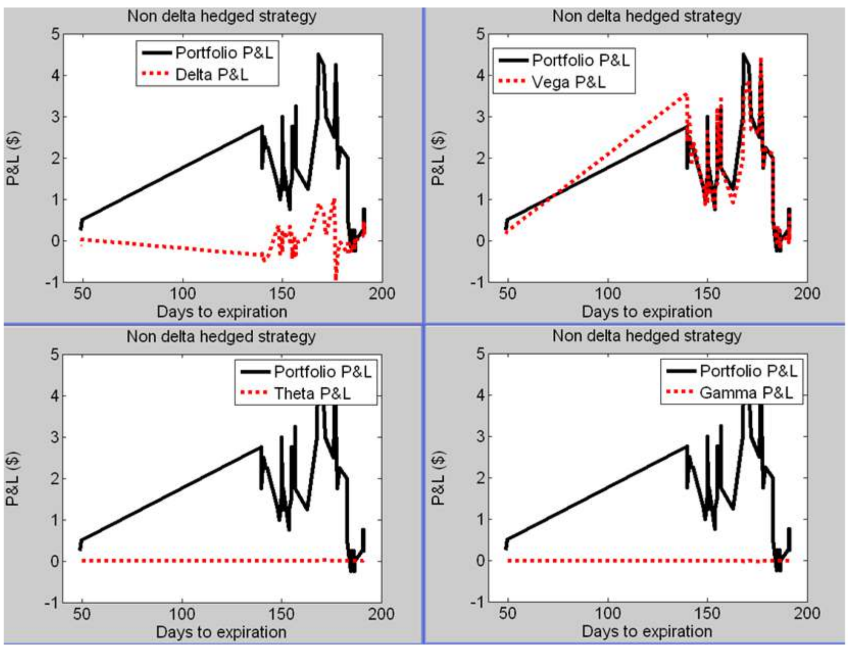

Figure 32: Contribution of each greek to the P&L of the portfolio, together with the total P&L. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

6.4.4 Strategy B, delta hedged

In figure 33 we show the P&L of each of the built portfolios, together with the greeks approximation .

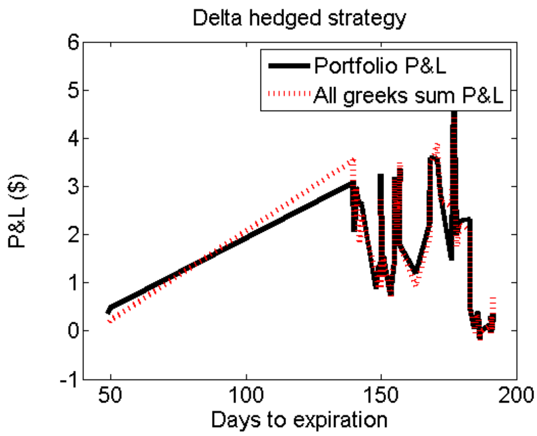

Figure 33: Total P&L, together with the summed greeks components, for every traded day. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

The total number of trades performed were 57 , from which 53 were winners.

In figure 34 we show the accrued P&L.

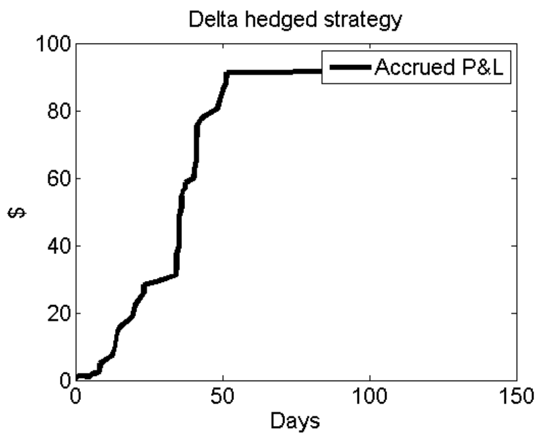

Figure 34: Contribution of each greek to the P&L of the portfolio, together with the total P&L. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

For each trade, we have calculated the contribution of each greek. This is shown in figure 35.

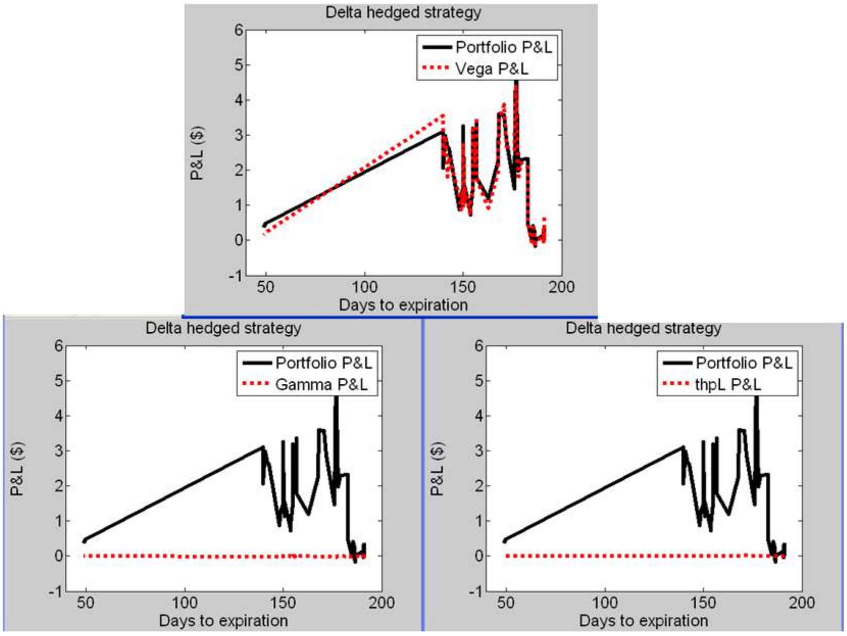

Figure 35: Contribution of each greek to the P&L of the portfolio, together with the total P&L. The P&L is given in units of the underlying, to obtain its actual value, it must be multiplied by the nominal of the contract

6.5 Strategy C

Our data did not show calls with $m = 80$. Therefore the performance of trade C could not be tested.

7 Conclusion and disccusion

According to figures 25, 28, 31, 34, the strategy is succesfull in all its versions. For the strategy A, perfoming delta hedging lead to a higher percentage of winning trades that the non delta hedge strategy (92.73% vs 75%). This is because the influence of delta is not neglige, compared to influence of vega (see figure 20). On the other hand, delta can contribute to losses, but also to earnings. For this reason, the strategy A delta hedged, has a lower accrued P&L than the non delta hedged. By non delta hedging the strategy, a non trivial amount of risk is added, which can lead to higher profits or losses.
Strategy B, can be performed less times than A, which means deep ITM calls are available in the market only when the time to expiration is large (more than 140 days, according to our data set). For the short period where this calls are available, Strategy B profits are higher than A profits. This is due to the fact that the vega sensitivity for large times to expiration is greater the deeper ITM are the call, for a wide range of strike prices (see figure 30)

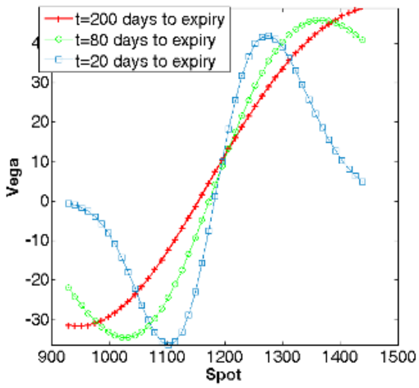

Figure 36: Vega sensitivity of a bearish call spread (strike prices: 1160, 1210. implied volatility: 30%) for three diferent times to expiry, 200, 80 and 20 days. For t=200, the deeper ITM are both calls, the greater vega, for abig range of strike prices)

Further studies should include the bid-ask spread in order to measure the real impact of the trading strategy.

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